The Volatility-Drag

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I review the impact on the future price level of random noise in the return process. This effect operates through the non-linearity of compounding. I look at how an increased level of volatility changes the price trajectories. Normally distributed (say) return variability lowers the median of the price distribution and skews it towards the upside. Over time, the skew increases as compounding operates. The majority of price trajectories are depressed by volatility. The mean price level is however unaffected as the lower trajectories are made up by a few 'explosive' ones. This will be important to bear in mind when discussing portfolio choice and the effect of portfolio rebalancing.

Volatility and compounding Consider a set of yearly returns (say) $(r_t)_{t=1,\ldots,T}$. The compounded return over the period $[0, T]$ is:

$$(1 + r_1) \cdots (1 + r_T).$$

If we want to assess the impact of the variability of returns, we might wish to compare this to $(1 + \bar{r})^T$, where:

$$\bar{r} = \frac{1}{T} \sum_{t=1}^{T} r_t.$$

Introducing the deviations to the mean $\varepsilon_t = r_t - \bar{r}$ $(t = 1, \ldots, T)$, we now need to compare:

$$(1 + \bar{r} + \varepsilon_1) \cdots (1 + \bar{r} + \varepsilon_T)$$

and $(1 + \bar{r})^T$. This is equivalent to comparing $1/T$ times the logs of these quantities, but then the inequality:

$$\frac{1}{T} \sum_{t=1}^{T} \log(1 + \bar{r} + \varepsilon_t) \leq \log(\frac{1}{T} \sum_{t=1}^{T} r_t) = \log(1 + \bar{r}),$$

is a simple consequence of the concavity of the log[1]. Given that the log is strictly

[1] For a concave function $f(\cdot)$ ($\mathbb{R} \to \mathbb{R}$), if $\sum_{i=1}^{n} \lambda_i x_i$ is a convex combination of $(x_i)_{i=1,\cdots,n}$, we have:

$$\sum_{i=1}^{n} \lambda_i f(x_i) \leq f(\sum_{i=1}^{n} \lambda_i x_i).$$

This property can be used to characterize concavity.

concave, we know that the inequality is strict unless the return series is constant. A simple second order approximation of the log function yields:

$$\frac{1}{T}\sum_{t=1}^{T}\log(1+\bar{r}+\varepsilon_t) \approx \log(1+\bar{r}) - \frac{1}{2}\frac{1}{(1+\bar{r})^2}Var(\varepsilon),$$

where $Var(\varepsilon)$ is the empirical variance of return shocks:

$$Var(\varepsilon) = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2.$$

The former relationship can be read as linking the geometric mean return and the arithmetic mean:

$$\left(\prod_{t=1}^{T}(1+r_t)\right)^{1/T} \approx \exp\left(\log(1+\bar{r}) - \frac{1}{2}\frac{1}{(1+\bar{r})^2}Var(\varepsilon)\right).$$

We thus see that variability in the returns $(r_t)_{t=1,\dots,T}$ lowers the compounded return. Of course, this result hinges on the anchoring of the *standard mean return*. In other words, the volatlity-drag is obtained when holding the average standard return constant and increasing the dispersion of realized standard returns around it. Variability in log-returns for a given mean log-return would have no effect on compounding[2]. The volatility-drag should not be over-interpreted. I now look at the consequences of volatility in more structured models of price dynamics, starting with the discrete time log normal model presented in this post.This will allow to identify the impact of volatility on the **distribution** of compounded returns and thereby better understand the volatility-drag. To simplify the description, I assume the expected returns $(r_t)_{t=1,\cdots,T}$ and volatilities $(\sigma_t)_{t=1,\cdots,T}$ are deterministic. In this setup, the compounded return over T periods is:

$$\prod_{t=1}^{T}\exp(r_t+\sigma_t\varepsilon_t-\frac{1}{2}\sigma_t^2) = \exp(\sum_{t=1}^{T}r_t+\sum_{t=1}^{T}\sigma_t\varepsilon_t-\sum_{t=1}^{T}\frac{1}{2}\sigma_t^2).$$

The median realization of $\sum_{t=1}^{T}\sigma_t\varepsilon_t$ is zero and under this scenario, the compounded return is just:

$$\exp(\sum_{t=1}^{T}r_t-\sum_{t=1}^{T}\frac{1}{2}\sigma_t^2).$$

The mean of the compounded return however is:

$$E_0\left[\exp(\sum_{t=1}^{T}r_t+\sum_{t=1}^{T}\sigma_t\varepsilon_t-\sum_{t=1}^{T}\frac{1}{2}\sigma_t^2)\right] = \exp(\sum_{t=1}^{T}r_t).$$

[2]One could therefore say that the volatility-drag is purely an artefact of the choice of units. Returns ar not measured in decibels, so to speak (decibels is a logarithmic scale for sound).

It is therefore apparent that volatility lowers the median compounded return but not the mean compounded return. Compared to the median, the mean is pushed upwards by the right tail of the distribution of shocks. This reflects the convexity of the exponential function! Volatility drives a constant wedge between the mean and the median. This has strong consequences over the long run. Assume t measures time in years, the annualized compounded return can be written:

$$\exp\left(\frac{1}{T}\left(\sum_{t=1}^{T} r_t - \sum_{t=1}^{T}\frac{1}{2}\sigma_t^2\right) + \frac{1}{T}\left(\sum_{t=1}^{T}\sigma_t\varepsilon_t\right)\right).$$

It is easy to design conditions such that as T goes to infinity:

$$\frac{1}{T}\left(\sum_{t=1}^{T} r_t - \sum_{t=1}^{T}\frac{1}{2}\sigma_t^2\right) \to c,$$

where c is a constant, and (law of large numbers) almost surely:

$$\frac{1}{T}\left(\sum_{t=1}^{T}\sigma_t\varepsilon_t\right) \to 0.$$

This means that in the long run, annualized returns get very close to the median return and the right tail of the annualized return distribution (i.e. the upside) becomes negligible. Volatility indeed drags down the annualized return with probability one in the long run. This suggests that a long term investor might choose a portfolio so as to maximize c. Although the annualized return converges to c, the set of non annualized compounded returns still contain trajectories which are much above cT as the volatility of the 'compounded' return shock:

$$\sum_{t=1}^{T}\sigma_t\varepsilon_t,$$

grows like:

$$\left(\sum_{t=1}^{T}\sigma_t^2\right)^{1/2} \propto \sqrt{T}.$$

From a finite horizon investment perspective finally, whether volatility is good or bad depends on how you weigh the different scenarios into your investment objective. It thus depends on the chosen utility function. When the utility function is logarithmic, the investor indeed maximizes c. This whole topic will require a specific development. Continuous time Unsurprisingly, the same reasoning can be carried out in continuous time using the standard geometric diffusions to model the price process:

$$\frac{dP_t}{P_t} = r_t dt + \sigma_t dW_t.$$

The only difference to the discrete case above is that sums are replaced by integrals. Conditions have to be designed such that for instance (law of large numbers)[3]:

$$\frac{1}{T}\left(\int_0^T \sigma_t dW_t\right) \to 0,$$

almost surely as T goes to infinity. When this is satisfied, the long run annualized return is characterized by:

$$\exp\left(\frac{1}{T}\left(\int_0^T r_t dt - \int_0^T \frac{1}{2}\sigma_t^2 dt\right)\right).$$

When comparing asset returns in the long run, one can then concentrate on the log of the above quantity, i.e.:

$$\frac{1}{T}\left(\int_0^T (r_t - \frac{1}{2}\sigma_t^2) dt\right).$$

Some authors (for example Fernholz[2002]) call $(r_t - \frac{1}{2}\sigma_t^2)$ the growth rate of the security although this might be a bit misleading. **Notes and references:** There is a literature that argues that investors should optimize the growth rate $(r_{p,t} - \frac{1}{2}\sigma_{p,t}^2)$ of a portfolio p. This is usually called the Kelly approach to investing. In the economic tradition, one starts with a utility function, and maximizes the utility of (say) terminal wealth. One can recover the Kelly investment rule by using a logarithmic utility function. More generally, optimal portfolios generated through a utility function contain a fraction of the Kelly optimal portfolio. For a recent review of the interaction between volatility and growth, see for instance Dempster[2007].

Dempster, M.A.H, Evstigneev, I.V., and Shenk-Hoppé, K.R. [2007], 'Volatility-Induced Financial Growth', *Quantitative Finance*, Vol 7 No. 2, April 2007, 151-260.

Fernholz, E.R. [2002], *Stochastic Portfolio Theory*, Springer.

Links

- Link to pdf

[3]In Fernholz[2002], section 1.3, Fernholz shows that the law of large numbers holds if $\lim_{t\to\infty} t^{-2}\left(\int_0^t \sigma_u^2 du\right)\log(\log(t)) = 0$. The proof of this fact hinges on the time changing technique and the law of interated logarithm for the Brownian motion.