Stochastic Integration and Localization

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As explained in this post, stochastic integration has initially been developed under the condition $E\left[\int_0^T H_s^2 d[M]_s\right] < \infty$*. This condition has then been replaced by the more general condition* $P\left(\int_0^T H_s^2 d[M]_s < \infty\right) = 1$*. This turns out to be the most general integrability condition under which stochastic integration can be defined. This post introduces the idea behind this generalization. The process whereby stochastic integration is thus generalized is called localization. In words, the above condition means that the set of trajectories for which integrated variance remains bounded should have probability one.*

Localizing stochastic integrals

This post on the stochastic integral emphasized the role of the quadratic variation in its definition. The quadratic variation of the integrator (a continuous martingale in effect) allowed to define a norm on the integrand space (some subspace of predictable processes, effectively incorporating a boundedness condition) so that a given suitable integrand could be approximated by simple predictable processes, the approximation process being controlled by the norm. For each simple predictable process, we have an explicit definition of the stochastic integral. The construction then consisted in making sure that as we get closer to the integrand using our simple processes the value of the integral stabilizes around a well defined random variable, which is our stochastic integral.

This construction is done for a given time interval $[0, T]$ and delivers a random variable $\int_0^T H_s dM_s$. The construction loses sight of the process obtained when time is allowed to vary. We would like to consider the trajectories $(\int_0^t H_s dM_s)_{t\in[0,T]}$ however. Fortunately, the process dimension can be recovered, and in our case (the integrator is a continuous martingale), $(\int_0^t H_s dM_s)_{t\in[0,T]}$ can be given continuous trajectories. Intuitively, the integral weighs the increments of the continuous martingale. The weighting scheme cannot break the continuity of the original martingale. This is easily verified for simple integrands. One needs to check that the continuity of the trajectories is preserved in the convergence process. It does. The proof of this result uses the fact that uniform limits of continuous functions are continuous.

The stochastic integral is initially defined under the boundedness condition:

$$E\left[\int_0^T H_s^2 d[M]_s\right] < \infty, \tag{*}$$

using L^2 Hilbert space techniques. The stochastic integral can still be defined if:

$$E\left[\int_0^T H_s^2 d[M]_s\right] = \infty,$$

provided:

$$P\left(\int_0^T H_s^2 d[M]_s < \infty\right) = 1.$$

One can then indeed identify an increasing sequence of sets of time and events which union is whole space $[0,T] \times \Omega$, and such that when restricted to each one of the sets in the sequence, the previous expectation (integral) is finite. This is done using stopping times $(T_k)_{k \in N}$, with each set of time and events being defined as $\{(t,\omega)|t \leq T_k(\omega)\}$. On each of these sets, the restricted integral is finite:

$$E\left[\int_0^{T_k} H_s^2 d[M]_s\right] < \infty.$$

Finally, the sequence of stopping times converges almost surely to T. In other words, the sequence of stopping times covers the time axis.

The set of stopping times is called a localizing sequence. It is used to define localized (stopped) versions of $(H_s)_{s \in [0,T]}$ and $(M_s)_{s \in [0,T]}$ through:

- $H_s^k = H_s$ on $\{(t,\omega)|t \leq T_k(\omega)\}$, $H_s^k = 0$ on $\{(t,\omega)|t\, T_k(\omega)\}$,
- $M_s^k = M_s$ on $\{(t,\omega)|t \leq T_k(\omega)\}$, $M_s^k = M_{T_k}$ on $\{(t,\omega)|t\, T_k(\omega)\}$.

The stochastic integral can then be defined for each stopped process (the right boundedness conditions having been ensured by construction) leading to a stochastic integral process which we can formally write down as $(\int_0^t H_s^k dM_s^k)_{t \in [0,T]}$. It can be shown for any two indices p and $q \geq p$:

$$\int_0^t H_s^q dM_s^q = \int_0^t H_s^p dM_s^p$$

on $\{(t,\omega)|t \leq T_p(\omega)\}$. One can thus glue all stopped integrals together to lead to an unambiguous stochastic integral[1]:

$$\int_0^t H_s dM_s,\ t \leq T.$$

[1]One can check that the construction is independent of the particular localizing sequence that has been chosen.

Stochastic integrals and local martingales

Relaxing the boundedness condition, we lose the property that the stochastic integral is a continuous martingale. It is instead a local continuous martingale. The concept of a local martingale is related to the previous construction. As above, the construct is designed to tame unboundedness. Essentially, a local martingale is a process coming with an increasing sequence of stopping times which fully covers the time axis and such that each stopped version of the process is a true martingale. Because in our construction above, the stopped processes verified the boundedness condition, each 'stopped stochastic integral' is a martingale. It is thus expected that our extended stochastic integral delivers a local martingale and it indeed does[2]. Finally, the integral can be fully generalized by allowing integrators to be local continuous martingales instead of true continuous martingales. First, local martingales have a proper continuous quadratic variation. Second, to adapt the above construction to this case, one needs to find a localizing sequence which both localizes the martingale and ensures the boundedness condition (*). This is possible and is actually easy in our case given the continuous trajectories of the local martingale.

In our context where integrators are continuous local martingales, the true constraint on extending stochastic integration is the divergence of $\int_0^T H_s^2 d[M]_s$ on a non negligible set of trajectories. Heuristically, when this quantity explodes, the stochastic integral oscillates wildly between $+\infty$ and $-\infty$ and its value cannot be defined. This corresponds to the asymptotic behavior of the Brownian motion at infinity ($\limsup_{t\to\infty} B_t = +\infty$, $\liminf_{t\to\infty} B_t = -\infty$). These wild oscillations have to remain the unreachable horizon of our stochastic integrals.

The moral of the story

The objective of this post was to introduce localization. A lot of details have been swept under the rug but hopefully, the logic is clear. Localization is the price we have to pay to be able to develop stochastic integration without imposing ad-hoc boundedness conditions. One can integrate local continuous martingales using predictable processes. The outcome is a local continuous martingale. The key condition which has to be enforced is that $\int_0^T H_s^2 d[M]_s$ be almost surely finite.

This post opens the door to the nirvana of (continuous) semimartingales. Semimartingales are precisely the processes for which stochastic integration makes sense. They decompose additively into finite variation processes and local martingales. We are one step away from semimartingales and we will remain there in this post. Using semimartingales, we will be able to explicit the rules of Ito

[2] A boundedness condition is needed to prove that a local martingale is a true martingale. Roughly speaking, the boundedness condition allows to apply the dominated convergence theorem of Lebesgue integration (or the related Vitali convergence theorem). As a reminder, the dominated convergence theorem identifies a sufficient condition for integration and limits to commute.

integration.

Links

- Link to pdf